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Stabilization for equations of one-dimensional viscous compressible heat-conducting media with nonmonotone equation of state

Bernard Ducomet[a,*] and Alexander Zlotnik[b,1]

[a] *Département de Physique Théorique et Appliquée, CEA/DAM Ile de France, BP 12, F-91680 Bruyères-le-Châtel, France*
[b] *Department of Mathematical Modelling, Moscow Power Engineering Institute, Krasnokazarmennaja 14, 111250 Moscow, Russia*

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Abstract

We consider the Navier–Stokes system describing motions of viscous compressible heat-conducting and "self-gravitating" media. We use the state function of the form $p(u,\theta) = p_0(u) + p_1(u)\theta$ linear with respect to the temperature θ, but we admit rather general nonmonotone functions p_0 and p_1 of u, which allows us to treat various physical models of nuclear fluids (for which p and u are the pressure and the specific volume) or thermoviscoelastic solids. For solutions to an associated initial-boundary value problem with "fixed–free" boundary conditions and arbitrarily large data, we prove a collection of estimates independent of time interval, including uniform two-sided bounds for u, and describe asymptotic behavior as $t \to \infty$. Namely, we establish the stabilization pointwisely and in L^q for u, in L^2 for θ, and in L^q for v (the velocity), for any $q \in [2, \infty)$. For completeness, the proof of the corresponding global existence theorem is also included.

Keywords: Stabilization; Large data; Viscous compressible heat-conducting fluid; Thermoviscoelastic solid; Nonmonotone equation of state

*Corresponding author. Fax: +33-1-69-26-70-63.
E-mail addresses: bernard.ducomet@cea.fr (B. Ducomet), zlotnik@apmsun.mpei.ac.ru (A. Zlotnik).
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1. Introduction

The problem of large-time behavior of solutions to equations of a one-dimensional (1d)-flow of viscous compressible heat-conducting fluids (or gases) with large data was studied in a lot of papers including [6,17,21,22,24]. All these papers deal with the case of particular (polytropic gas) or general pressure law $p(u,\theta)$ but always *monotone* (decreasing) with respect to the variable u; here u and θ are the specific volume and the absolute temperature. It is well known that this monotonicity is not valid in a number of physical situations. In particular, the case of the two-term pressure

$$p(u,\theta) = p_0(u) + p_1(u)\theta, \tag{1}$$

which is linear in θ but with complicated nonmonotone $p_0(u)$ is of importance for nuclear fluid models, see [9–11] and references therein.

The case of a *nonmonotone* response function (1) with other properties of p_0 and p_1, and nonmonotone p_1 is also important in the completely different physical context of thermoviscoelasticity, where the relaxation of monotonicity can model new physical phenomena of phase transition type leading to interesting technological applications, one of these concerning the so-called shape memory alloys (see [28] and references therein).

In past years, a number of works have addressed the problem of global existence for such thermoviscoelastic models [7,8,16,30,31]. In related papers, for models with essentially simplified forms of the viscosity term and heat flux in the equations, the stabilization of solutions was also studied under various boundary conditions [14,15,25–27] but for u it was proved only in the restricted case $p_0 = 0$. Let us also mention papers concerning stabilization in nonmonotone barotropic case (where $p = p(u)$) for fluids [12,13,19,34] and for viscoelastic solids [5,23].

Note that nonmonotonicity of p complicates in an essential way the problem of stabilization. In particular, the stationary specific volume becomes nonunique and can be discontinuous.

In this paper, we consider the pressure law (1) with rather general nonmonotone p_0 and p_1 and we study both the cases of nuclear fluids and of thermovisco-elastic solids (without the aforementioned simplification in the viscosity term and the heat flux). Moreover a large external force of "self-gravitation" type is also taken into consideration. For an initial-boundary value problem with "fixed–free" boundary conditions and large initial data, we prove a collection of estimates independent of time interval for solutions, including two-sided bounds for the specific volume u. Moreover we establish the pointwise and L^q-stabilization for u, L^2-stabilization for the temperature θ and the pressure p, and L^q-stabilization for the velocity for any $q \in [2,\infty)$, as time tends to infinity. In the nuclear fluid case, we also justify the sharpness of the main condition on the "self-gravitation" force.

2. Statement of the problem and main results

We consider the following system of quasilinear differential equations for 1d-motions of viscous compressible heat-conducting media

$$\begin{cases} u_t = v_x, \\ v_t = \sigma_x + g, \\ e[u,\theta]_t = \sigma v_x + \pi_x, \end{cases} \tag{2}$$

where $(x,t) \in Q \equiv \Omega \times \mathbf{R}^+ = (0, M) \times (0, \infty)$ are the Lagrangian mass coordinates, with M being the total mass of the medium.

The unknown quantities $u > 0$, v, and $\theta > 0$ are the specific volume, the velocity, and the absolute temperature. We also denote by $\rho = \frac{1}{u}$ the density, $\sigma = \nu \rho v_x - p[u,\theta]$ the stress, $e(u,\theta)$ the internal energy, and $-\pi = -\kappa[u,\theta]\rho\theta_x$ the heat flux. Hereafter the notation $\lambda[u,\theta](x,t) = \mu(u(x,t),\theta(x,t))$, for $\lambda = e, p, \kappa$, etc. is adopted.

In order to fix the state functions $p(u,\theta)$ and $e(u,\theta)$, we define the Helmholtz free energy

$$\Psi(u,\theta) = -c_V \theta \log \theta - P_0(u) - P_1(u)\,\theta,$$

where $c_V = \text{const} > 0$. Then thermodynamics tells us that

$$p(u,\theta) = -\Psi_u(u,\theta) = p_0(u) + p_1(u)\theta, \tag{3}$$

with $p_0 = P_0'$ and $p_1 = P_1'$, as well as

$$e(u,\theta) = \Psi(u,\theta) - \theta\Psi_\theta(u,\theta) = -P_0(u) + c_V\theta, \tag{4}$$

where $\Psi_u = \frac{\partial \Psi}{\partial u}$ and $\Psi_\theta = \frac{\partial \Psi}{\partial \theta}$.

First, we consider the more difficult case of the nuclear fluid. We suppose that the functions $p_0, p_1 \in C^1(\mathbf{R}^+)$ are such that[2]

$$\lim_{u \to 0^+} p_0(u) = +\infty, \quad \lim_{u \to +\infty} p_0(u) = 0, \tag{5}$$

$$p_1(u) \geqslant 0, \quad u\,p_1(u) = O(1) \quad \text{as } u \to +\infty. \tag{6}$$

Suppose also that the viscosity and heat conductivity coefficients are such that $\nu = \text{const} > 0$ and $\kappa \in C^1(\mathbf{R}^+ \times \mathbf{R}^+)$, with $0 < \underline{\kappa} \leqslant \kappa(u,\theta) \leqslant \bar{\kappa}$, where $\underline{\kappa}$ and $\bar{\kappa}$ are given constants. We do not impose any growth conditions on the derivatives of κ.

The so-called "self-gravitation force" $g \in L^1(\Omega)$ is taken into account. In fact, this name does not correspond exactly to the physical situation, as, at least in the nuclear fluid case, the corresponding "physical" force is the Coulomb force between charged

[2] Note that $C^1(\mathbf{R}^+)$ stands for the space of continuously differentiable functions on $\mathbf{R}^+$, but not necessarily bounded. The spaces $C^1(\mathbf{R}^+ \times \mathbf{R}^+)$, $C(\mathbf{R}^+)$, $C(\mathbf{R})$, etc. used below are understood similarly.

particles, which contrary to the Newton gravitational force, is attractive. Although the distinction Coulomb–Newton is of utmost importance in multidimensional problems, it is harmless in the 1d-context.

We supplement Eqs. (2) with the following boundary and initial conditions:

$$v|_{x=0} = 0, \quad \sigma|_{x=M} = -p_\Gamma, \quad \theta|_{x=0} = \theta_\Gamma, \quad \pi|_{x=M} = 0, \tag{7}$$

$$u|_{t=0} = u^0(x), \quad v|_{t=0} = v^0(x), \quad \theta|_{t=0} = \theta^0(x), \tag{8}$$

with an outer pressure $p_\Gamma = \text{const}$ and a given temperature $\theta_\Gamma = \text{const} > 0$.

From a physical point of view, our dynamical boundary conditions correspond to a free-boundary problem: we impose a fixed stress on the right boundary (fixed external pressure in the fluid context or stress-free condition for $p_\Gamma = 0$ in the solid context) and consider the fixed left boundary. For the thermal boundary conditions, we suppose that the temperature is known on the fixed boundary and the flux is zero on the free one.

Throughout the paper, we use the classical Lebesgue spaces $L^q(G)$ together with their anisotropic version $L^{q,r}(Q)$, for $q, r \in [1, \infty]$, and we denote the associated norm by $\|\cdot\|_{L^{q,r}(Q)} = \|\, \|\cdot\|_{L^q(\Omega)} \|_{L^r(\mathbf{R}^+)}$. In Section 3, we also use the abbreviation $\|\cdot\|_G$ for $\|\cdot\|_{L^2(G)}$. Let also $V_2(Q)$ [20] be the standard space of functions w having finite (parabolic) energy $\|w\|_{V_2(Q)} = \|w\|_{L^{2,\infty}(Q)} + \|w_x\|_{L^2(Q)}$. We denote by $H^1(\Omega)$ (resp. $H^{2,1}(Q_T)$) the standard Sobolev space equipped with the norm $\|\varphi\|_{H^1(\Omega)} = \|\varphi\|_{L^2(\Omega)} + \|\varphi_x\|_{L^2(\Omega)}$ (resp. $\|w\|_{H^{2,1}(Q_T)} = \|w\|_{L^{2,\infty}(Q_T)} + \|w_x\|_{V_2(Q_T)} + \|w_t\|_{L^2(Q_T)}$). Hereafter $Q_T = \Omega \times (0, T)$.

In Section 3 and the appendix, we shall also exploit the integration operators $I^*\varphi(x) = \int_x^M \varphi(\xi)\, d\xi$, for $\varphi \in L^1(\Omega)$, and $I_0 a(t) = \int_0^t a(\tau)\, d\tau$, for $a \in L^1(0, T)$.

Suppose that the initial data are such that $u^0 \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega u^0 > 0$, $v^0 \in L^4(\Omega)$, $\theta^0 \in L^2(\Omega)$, $\log \theta^0 \in L^1(\Omega)$ with $\theta^0 > 0$. Though it is possible to establish our main results for weak solutions [1], to simplify the presentation, we restrict ourselves to the case of so-called regular weak (or strong) solutions [6] such that $u \in L^\infty(Q_T)$, $u_x, u_t \in L^{2,\infty}(Q_T)$, $\min_{\bar{Q}_T} u > 0$, and $v, \theta \in H^{2,1}(Q_T)$, $\min_{\bar{Q}_T} \theta > 0$ for any $T > 0$. We consider the problem of existence of the latter solutions in the appendix.

Now we summarize our main results concerning the problem (2), (7), (8), under conditions (5),(6). Let us define the function

$$p_S(x) := p_\Gamma - \int_x^M g(\xi)\, d\xi \quad \text{for } x \in \bar{\Omega},$$

which plays the role of a stationary pressure, and set $\underline{p}_S := \min_{\bar{\Omega}} p_S$ and $\bar{p}_S := \max_{\bar{\Omega}} p_S$. Obviously $\underline{p}_S \leqslant p_\Gamma \leqslant \bar{p}_S$. Let $N > 1$ be an arbitrarily large parameter and $K_i = K_i(N)$ and $K^{(i)} = K^{(i)}(N)$, $i = 0, 1, 2, \ldots$, be positive nondecreasing functions of N, which can also depend on $M, \nu, \underline{\kappa}, \bar{\kappa}$, etc, but neither on the initial data nor on g.

Theorem 1. (1) *Suppose that the initial data*, p_Γ, *and* g *are such that*

$$N^{-1} \leqslant u^0 \leqslant N, \quad ||v^0||_{L^4(\Omega)} + ||\log \theta^0||_{L^1(\Omega)} + ||\theta^0||_{L^2(\Omega)} \leqslant N, \tag{9}$$

$$||g||_{L^1(\Omega)} \leqslant N, \quad N^{-1} \leqslant \underline{p}_S. \tag{10}$$

Then the following estimates in Q *together with* $L^2(\Omega)$-*stabilization property hold*

$$0 < K_1^{-1} = \underline{u} \leqslant u(x,t) \leqslant \bar{u} = K_2 \quad \text{in } \bar{Q}, \tag{11}$$

$$\begin{aligned} &||v||_{V_2(Q)} + ||v^2||_{V_2(Q)} + ||\log \theta||_{L^{1,\infty}(Q)} + ||(\log \theta)_x||_{L^2(Q)} \\ &\quad + ||\theta - \theta_\Gamma||_{V_2(Q)} \leqslant K_3, \\ &\quad ||p[u,\theta] - p_S||_{L^2(Q)} \leqslant K_4, \\ &\quad ||v^2(\cdot,t)||_{L^2(\Omega)} + ||\theta(\cdot,t) - \theta_\Gamma||_{L^2(\Omega)} + ||p[u,\theta](\cdot,t) - p_S(\cdot)||_{L^2(\Omega)} \to 0 \\ &\quad \text{as } t \to \infty. \end{aligned} \tag{12}$$

(2) *Suppose that* $p(u,\theta)$ *satisfies the following additional condition*:

$$\begin{aligned} &\text{For any } c \in [\underline{p}_S, \bar{p}_S], \text{ there exists no interval } (u_1,u_2) \text{ such that} \\ &p(u,\theta_\Gamma) \equiv c \quad \text{on } (u_1,u_2). \end{aligned} \tag{13}$$

Then the following pointwise stabilization property holds for u: *there exists a function* $u_S \in L^\infty(\Omega)$ *satisfying*

$$p(u_S(x),\theta_\Gamma) = p_S(x) \quad \text{and} \quad \underline{u} \leqslant u_S(x) \leqslant \bar{u} \quad \text{on } \bar{\Omega}, \tag{14}$$

such that

$$u(x,t) \to u_S(x) \quad \text{as } t \to \infty, \text{ for all } x \in \bar{\Omega}, \tag{15}$$

and consequently $||u(\cdot,t) - u_S(\cdot)||_{L^q(\Omega)} \to 0$ *as* $t \to \infty$, *for any* $q \in [1,\infty)$.

(3) *Suppose that*, *additionally to the hypotheses of Claim* 1, $||v^0||_{L^q(\Omega)} \leqslant N$, *for some* $q \in (4,\infty)$. *Then the following estimate in* Q *together with* $L^q(\Omega)$-*stabilization property hold*:

$$||v||_{L^{q,\infty}(Q)} + ||v||_{L^{\infty,q}(Q)} \leqslant K_5 q,$$

$$||v(\cdot,t)||_{L^q(\Omega)} \to 0 \quad \text{as } t \to \infty,$$

where K_5 *does not depend on* q.

Remark 1. (1) The second condition (10) imply that $N^{-1} \leqslant p_\Gamma$.

(2) An elementary but important consequence of Claim 2 is that $V(t) := \int_\Omega u(x,t)\,dx \to V_S > 0$ as $t \to \infty$, where $V(t)$ is the volume of the fluid (or in other words, the Eulerian position of the free boundary).

(3) For nonmonotone $p(u, \theta_\Gamma)$, if there exist two points $0 < u^{(1)} < u^{(2)}$ such that

$$\underline{p}_S < p^{(1)} := p(u^{(1)}, \theta_\Gamma) < p^{(2)} := p(u^{(2)}, \theta_\Gamma) < \bar{p}_S,$$

and moreover

$$\begin{cases} p^{(1)} \leqslant p(u, \theta_\Gamma), & \text{for } 0 < u \leqslant u^{(1)}, \\ p^{(1)} \leqslant p(u, \theta_\Gamma) \leqslant p^{(2)}, & \text{for } u^{(1)} < u < u^{(2)}, \\ p(u, \theta_\Gamma) \leqslant p^{(2)}, & \text{for } u^{(2)} \leqslant u, \end{cases}$$

then necessarily $u_S \notin C(\bar{\Omega})$. Moreover, consequently, the convergence in (15) cannot be uniform in x. In fact, even for $g \equiv 0$, if the equation $p(u, \theta_\Gamma) = p_\Gamma$ has more than one solution, then u_S can be discontinuous in $\bar{\Omega}$. Namely, if this equation has exactly k solutions $u^{(1)} < \cdots < u^{(k)}$, then the function u_S can be written as

$$u_S = \sum_{j=1}^{k} \chi(E_j)\, u^{(j)},$$

where $E_j, 1 \leqslant j \leqslant k$, are any measurable nonintersecting subsets of $\bar{\Omega}$ (some of them may be empty) such that $\cup_{j=1}^k E_j = \bar{\Omega}$, and $\chi(E_j)$ are their characteristic functions. Unfortunately, we cannot assert more about u_S.

Let us justify that the second condition (10) is essential in Theorem 1. Set $m(\theta_\Gamma) := \inf_{u>0} p(u, \theta_\Gamma)$. Obviously $m(\theta_\Gamma) \leqslant 0$, and if $p_0 \geqslant -p_1\theta_\Gamma$, then $m(\theta_\Gamma) = 0$.

Proposition 1. *Let the hypotheses of Theorem* 1, *Claim* 1, *be valid, but suppose that* $\underline{p}_S < m(\theta_\Gamma)$, *instead of* $N^{-1} \leqslant \underline{p}_S$. *Then*

$$\limsup_{t \to \infty} V(t) = \infty. \tag{16}$$

This property means that the upper bound for u in (11) is violated and physically, that the fluid can asymptotically expand in the whole halfspace.

Let us also consider the borderline case $\underline{p}_S = m(\theta_\Gamma)$.

Proposition 2. *Let the hypotheses of Theorem* 1, *Claim* 1, *be valid and* $p(u, \theta_\Gamma) > m(\theta_\Gamma) = 0$, *but* $p_S(0) = 0$ *instead of* $N^{-1} \leqslant \underline{p}_S$. *Then at least one of the* *following properties holds*:

$$\limsup_{t\to\infty}\left|\int_\Omega v(x,t)\,dx\right| = \infty, \tag{17}$$

$$\lim_{t\to\infty} u(0,t) = \infty. \tag{18}$$

If in addition $\int_1^\infty p(u,\theta_\Gamma)\,du<\infty$ *and* $\underline{p}_S = p_S(0) = 0$, *then* $\|v\|_{L^{2,\infty}(Q)}\leqslant K_3$ *whereas property* (18) *holds*.

Properties (17) and (18) mean that estimate (12) and the upper bound for u in (11) are violated, respectively. Note that Propositions 1 and 2 go back to results of [34] where the barotropic case was studied.

Finally, we turn to the case of thermoviscoelastic solids. Let the above introduced $\underline{p}_S\leqslant\bar{p}_S$ be fixed. Suppose that, instead of (5) and (6), the following conditions hold

$$\bar{p}_S\leqslant p_0(u) \quad \text{and} \quad 0\leqslant p_1(u) \quad \text{for } 0<u\leqslant\check{u}, \tag{19}$$

$$p_0(u)\leqslant\underline{p}_S \quad \text{and} \quad p_1(u)\leqslant 0 \quad \text{for } 0<\hat{u}\leqslant u, \tag{20}$$

for some $0<\check{u}\leqslant\hat{u}<\infty$. The conditions of such kind are of standard type for the thermoviscoelastic case.

Theorem 2. *All Claims* 1–3 *of Theorem* 1 *remain valid under conditions* (19) *and* (20), *and without the condition* $N^{-1}\leqslant\underline{p}_S$.

Remark 2. We could consider the viscosity coefficient $v = v(u)\geqslant v_0>0$, $v\in C^1(\mathbf{R}^+)$ as well as body force and boundary data in the form $g(x,t) = g_S(x) + \Delta g(x,t)$, $p_\Gamma(t) = p_{\Gamma,S} + \Delta p_\Gamma(t)$, and $\theta_\Gamma(t) = \theta_{\Gamma,S} + \Delta\theta_\Gamma(t)$, with perturbations $\Delta g, \Delta p_\Gamma$, and $\Delta\theta_\Gamma$ tending to zero as $t\to\infty$ in some weak sense (cf. the barotropic case [13,34]). To simplify the presentation of the results and their proof, we do not realize this possibility in the paper.

3. Proof of the results

We begin with the proof of Theorem 1 which follows from a lengthy series of lemmas, providing necessary a priori estimates and stabilization properties: Claims 1, 2, and 3 will be proved respectively in Lemmas 1–9, Lemmas 10 and 11, and Lemmas 12 and 13.

Then we proceed with the proofs of Propositions 1 and 2 and Theorem 2.

3.1. A priori estimates and proof of Theorem 1

Lemma 1. *The following energy estimates hold*:

$$||u||_{L^{1,\infty}(Q)} + ||v||_{L^{2,\infty}(Q)} + ||\theta||_{L^{1,\infty}(Q)} + ||\log\theta||_{L^{1,\infty}(Q)} \leqslant K^{(1)}, \tag{21}$$

$$\left|\left|\sqrt{\frac{\rho}{\theta}}v_x\right|\right|_Q + \left|\left|\frac{\sqrt{\rho}}{\theta}\theta_x\right|\right|_Q \leqslant K^{(2)}. \tag{22}$$

Proof. Eqs. (2) and (3), (4) imply the equations

$$\left(\frac{1}{2}v^2 + e[u,\theta]\right)_t = (\sigma v + \pi)_x + gv, \tag{23}$$

$$c_V\theta_t = \pi_x + (v\rho v_x - p_1[u]\theta)v_x. \tag{24}$$

Hereafter we use the notation $\lambda[u](x,t) = \lambda(u(x,t))$, for $\lambda = p_i, P_i,\ i = 0, 1$, etc.

By multiplying the second equation by $\frac{\theta_\Gamma}{\theta}$ and subtracting the result from the first one, we obtain

$$\begin{aligned}&\left(\frac{1}{2}v^2 + e[u,\theta] - c_V\theta_\Gamma\log\frac{\theta}{\theta_\Gamma} - \theta_\Gamma P_1[u] + p_\Gamma u\right)_t + \theta_\Gamma v\frac{\rho}{\theta}v_x^2\\ &\quad = ((\sigma + p_\Gamma)v)_x + \left(1 - \frac{\theta_\Gamma}{\theta}\right)\pi_x + gv.\end{aligned}$$

By setting $P(u,\theta) := P_0(u) + P_1(u)\,\theta$, integrating this equality over Ω, and using the formula

$$\int_\Omega gv\,dx = \int_\Omega (I^*g)v_x\,dx = \frac{d}{dt}\int_\Omega (I^*g)u\,dx,$$

we finally get, for any constant C,

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left[\frac{1}{2}v^2 + c_V\theta_\Gamma\left(\frac{\theta}{\theta_\Gamma} - \log\frac{\theta}{\theta_\Gamma}\right) + p_S u - P[u,\theta_\Gamma] + C\right]dx\\ &\quad + \theta_\Gamma\int_\Omega\left(v\frac{\rho}{\theta}v_x^2 + \kappa[u,\theta]\frac{\rho}{\theta^2}\theta_x^2\right)dx = 0.\end{aligned} \tag{25}$$

Conditions (5) and (6) imply the property

$$P(u,\theta_\Gamma) \leqslant \varepsilon u + C_\varepsilon \quad \text{on } \mathbf{R}^+,\ \ \forall\varepsilon > 0.$$

By integrating (25) over $(0, T)$ for any $T > 0$, applying conditions (9) and (10), and choosing $\varepsilon := \frac{1}{2}\underline{p}_S$, we obtain estimates (21) and (22). Here, the elementary inequality $\frac{1}{2}\alpha \leqslant \alpha - \log\alpha + \log 2 - 1$ is taken into account. $\square$

The following auxiliary result on ordinary differential inequalities is useful to prove lower and upper bounds for the specific volume u in various situations.

Lemma 2. *Let $N_0 \geqslant 0$, $N_1 \geqslant 0$, and $\varepsilon_0 > 0$ be three parameters.*

Let $f \in C(\mathbf{R})$ and $y, b \in W^{1,1}(0, T)$, for any $T > 0$. The following claims are valid:

(1) *if*

$$\frac{dy}{dt} \geqslant f(y) + \frac{db}{dt} \quad on\ \mathbf{R}^+,$$

where $f(-\infty) = +\infty$ and $b(t) - b(\tau) \geqslant -N_0 - N_1(t - \tau)$, for any $0 \leqslant \tau \leqslant t$, then the uniform lower bound holds:

$$\min\{y(0), \check{z}\} - N_0 \leqslant y(t) \quad on\ \bar{\mathbf{R}}^+,$$

where $\check{z} = \check{z}(N_1)$ is such that $f(z) \geqslant N_1$, for $z \leqslant \check{z}$;

(2) *if*

$$\frac{dy}{dt} \leqslant f(y) + \frac{db}{dt} \quad on\ \mathbf{R}^+,$$

where $\limsup_{z \to +\infty} f(z) \leqslant 0$ *and $b(t) - b(\tau) \leqslant N_0 - \varepsilon_0(t - \tau)$, for any $0 \leqslant \tau \leqslant t$, then the uniform upper bound holds:*

$$y(t) \leqslant \max\{y(0), \hat{z}\} + N_0 \quad on\ \bar{\mathbf{R}}^+,$$

where $\hat{z} = \hat{z}(\varepsilon_0)$ is such that $f(z) \leqslant \varepsilon_0$, for $z \geqslant \hat{z}$.

Remark 3. In Lemma 2, one can drop the conditions $f(-\infty) = +\infty$ and $\limsup_{z \to +\infty} f(z) \leqslant 0$, take $f \in C(\mathbf{R} \times \bar{\mathbf{R}}^+)$ and replace $f(y)$ by $f(y, t)$. Then Claim 1 remains valid provided that, for a fixed N_1, there exists $\check{z}$ such that $f(z, t) \geqslant N_1$, for $z \leqslant \check{z}$ and $t \geqslant 0$. Similarly, Claim 2 remains valid provided that, for a fixed $\varepsilon_0 \in \mathbf{R}$, there exists $\hat{z}$ such that $f(z, t) \leqslant \varepsilon_0$, for $z \geqslant \hat{z}$ and $t \geqslant 0$.

Lemma 2 is borrowed from [33], where in both claims, differential equalities are used, but one checks easily that the proof remains valid for inequalities; the similar conclusion is valid concerning Remark 3. The modified claims of the type specified in this remark are well known in viscoelastic and thermoviscoelastic contexts.

Lemma 3. *The uniform lower bound holds:* $0 < \underline{u} = (K^{(3)})^{-1} \leqslant u(x, t)$ *in* $\bar{Q}$.

Proof. The action of the operator I^* on the second equation (2) gives the equation

$$I^* v_t = -\nu\rho v_x + p[u,\theta] - p_S, \tag{26}$$

which together with the relation $\rho v_x = (\log u)_t$ lead to the another important equation

$$(\nu \log u)_t = p[u,\theta] - p_S - I^* v_t. \tag{27}$$

By putting $y := \nu \log u$, exploiting the property $p_1[u]\theta \geqslant 0$, and fixing any $x \in \bar{\Omega}$, we get

$$\frac{dy}{dt} \geqslant p_0\left(\exp\frac{y}{\nu}\right) - \bar{p}_S - \frac{d}{dt} I^* v.$$

The function $f(z) := p_0(\exp\frac{z}{\nu}) - \bar{p}_S$ has the property $f(-\infty) = +\infty$, see (5). Moreover, due to the energy estimate (21)

$$|I^* v|_\tau^t| \leqslant 2 \sup_{\bar{Q}} |I^* v| \leqslant 2M^{1/2} ||v||_{L^{2,\infty}(Q)} \leqslant K_0. \tag{28}$$

Now Claim 1 in Lemma 2 (with $N_1 = 0$) implies the estimate

$$\min\{\nu \log u^0(x), \nu \log \check{u}\} - K_0 \leqslant y(x,t),$$

with a number $\check{u}$ such that $p_0(u) - \bar{p}_S \geqslant 0$, for any $0 < u \leqslant \check{u}$. Then:

$$\underline{u} := \min\{N^{-1}, \check{u}\} \exp\left(-\frac{K_0}{\nu}\right) \leqslant u(x,t) \quad \text{in } \bar{Q}. \qquad \square$$

The next auxiliary result on ordinary integral inequality is useful to deduce a uniform upper bound for u.

Lemma 4. *Let b be a nondecreasing function on $[0,T]$ with $b(0) \geqslant 0$, and let $a \in L^1(0,T)$ be a nonnegative function. If $z \in L^\infty(0,T)$, $z \geqslant 0$ satisfies*

$$z(t) \leqslant b(t) + \int_0^t a(\tau) z(\tau)\, d\tau \quad \text{on } (0,T),$$

then the upper bound holds:

$$z(t) \leqslant b(t) \exp \int_0^t a(\tau)\, d\tau \leqslant b(t) \exp ||a||_{L^1(0,T)} \quad \text{on } (0,T).$$

The result follows immediately from the integral Gronwall's Lemma (for example see [6]) if one takes into account that

$$z(s) \leqslant b(t) + \int_0^s a(\tau) z(\tau)\, d\tau \quad \text{for } 0 < s \leqslant t < T.$$

Lemma 5. *The uniform upper bound holds*: $u(x,t) \leqslant \bar{u} = K^{(4)}$ *in* $\bar{Q}$.

Proof. Let us rewrite the first equation (2) as follows

$$u_t = \frac{1}{v}(\sigma + \delta)\ u + \frac{1}{v}u(p[u,\theta] - \delta),$$

where δ is a parameter. We consider this equation as an ordinary differential one with respect to u and obtain the formula

$$u = \exp\left(\frac{1}{v}I_0(\sigma+\delta)\right)\left\{u^0 + \frac{1}{v}I_0\left[\exp\left(-\frac{1}{v}I_0(\sigma+\delta)\right)u(p[u,\theta]-\delta)\right]\right\}. \qquad (29)$$

By applying the operator I_0 to Eq. (26), we find

$$I_0\sigma = -p_S t - I^*(v - v^0).$$

Thus by choosing $\delta := \frac{1}{2}\underline{p}_S$ and using estimate (28), we get

$$\frac{1}{v}I_0(\sigma+\delta)|_\tau^t = -\frac{1}{v}(p_S - \delta)(t-\tau) - \frac{1}{v}I^*v|_\tau^t \leqslant -\alpha(t-\tau) + K_1 \quad \text{on } \bar{\Omega},$$

for all $0 \leqslant \tau \leqslant t$, with $\alpha := \frac{1}{2v}\underline{p}_S > 0$. Conditions (5) and (6) on p_0 and p_1 together with the lower bound $\underline{u} \leqslant u$ give

$$u(p[u,\theta]-\delta) \leqslant u\max\{p_0[u]-\delta, 0\} + up_1[u]\theta \leqslant K_2 + K_3\theta.$$

Therefore formula (29) implies the estimate

$$\hat{u}(t) := ||u(\cdot,t)||_{L^\infty(\Omega)} \leqslant K_4 e^{-\alpha t}\left[1 + \int_0^t e^{\alpha\tau}(1 + ||\theta(\cdot,\tau)||_{L^\infty(\Omega)})\,d\tau\right]. \qquad (30)$$

We set $a := |||\frac{\sqrt{\rho}}{\theta}\theta_x||_\Omega^2$. It is well known [2,6] that the inequalities

$$||\theta||_{L^\infty(\Omega)} \leqslant \theta_\Gamma + ||\theta_x||_{L^1(\Omega)} \leqslant \theta_\Gamma + (a||\theta||_{L^1(\Omega)}||\theta||_{L^\infty(\Omega)}\hat{u})^{1/2}$$
$$\leqslant \varepsilon||\theta||_{L^\infty(\Omega)} + \theta_\Gamma + \frac{1}{4\varepsilon}a||\theta||_{L^1(\Omega)}\hat{u} \quad \forall \varepsilon > 0$$

together with the estimate $||\theta||_{L^{1,\infty}(Q)} \leqslant K^{(1)}$ imply

$$||\theta||_{L^\infty(\Omega)} \leqslant K_5(1 + a\hat{u}).$$

Thus by using estimate (30), the function $z(t) := e^{\alpha t}\hat{u}(t)$ satisfies

$$z(t) \leqslant K_6\left(e^{\alpha t} + \int_0^t a(\tau)z(\tau)\,d\tau\right) \quad \text{on } \mathbf{R}^+.$$

Since $||a||_{L^1(\mathbf{R}^+)} \leqslant (K^{(2)})^2$ according to Lemma 1, by using Lemma 4

$$z(t) \leqslant K_6 \exp(\alpha t + K_6(K^{(2)})^2) = K^{(4)}e^{\alpha t} \quad \text{on } \mathbf{R}^+.$$

This means that $u \leqslant \hat{u} \leqslant \bar{u} := K^{(4)}$ in $\bar{Q}$. $\square$

Corollary 1. *For v, the following estimate holds*:

$$\frac{1}{\sqrt{M}}||v||_Q \leqslant ||v||_{L^{\infty,2}(Q)} \leqslant (K^{(1)})^{1/2}\left|\left|\frac{v_x}{\sqrt{\theta}}\right|\right|_Q \leqslant K^{(5)}.$$

Proof. In fact, by using Lemma 1, we have

$$||v||_{C(\bar{\Omega})} \leqslant ||v_x||_{L^1(\Omega)} \leqslant ||\theta||^{1/2}_{L^1(\Omega)}\left|\left|\frac{v_x}{\sqrt{\theta}}\right|\right|_\Omega \leqslant (K^{(1)})^{1/2}\left|\left|\frac{v_x}{\sqrt{\theta}}\right|\right|_\Omega \tag{31}$$

and

$$\left|\left|\frac{v_x}{\sqrt{\theta}}\right|\right|_Q \leqslant \bar{u}^{\,1/2}\left|\left|\sqrt{\frac{\rho}{\theta}}v_x\right|\right|_Q \leqslant \bar{u}^{\,1/2}K^{(2)}. \qquad \square$$

Note that similarly $||(\log\theta)_x||_Q \leqslant \bar{u}^{1/2}K^{(2)}$.

The following auxiliary result on ordinary differential inequalities will be exploited when proving $V_2(Q)$-estimates and $L^2(\Omega)$-stabilization for v^2 and $\theta - \theta_\Gamma$.

Lemma 6. *Let $a_0 = \text{const} > 0$ and $a, h \in L^1(\mathbf{R}^+)$. If a function $y \geqslant 0$ on $\mathbf{R}^+$ satisfies $y \in W^{1,1}(0,T)$ for any $T > 0$ and*

$$\frac{dy}{dt} + (a_0 + a)y \leqslant h \quad \textit{on } \mathbf{R}^+, \tag{32}$$

then the following upper bound together with stabilization property hold:

$$\begin{aligned} &\sup_{t \geqslant 0} y(t) \leqslant (y(0) + ||h||_{L^1(\mathbf{R}^+)})\exp||a||_{L^1(\mathbf{R}^+)}, \\ &y(t) \to 0 \quad \textit{as } t \to \infty. \end{aligned} \tag{33}$$

This simple known result is easily derivable by multiplying (32) by $\exp I_0(a_0 + a)$ and integrating the result; of course estimate (33) holds also for $a_0 = 0$. Note that more general result can be found in [29, Lemma 2.1].

Lemma 7. *For v^2 and $\theta-\theta_\Gamma$, the following estimate together with the stabilization property hold*:

$$\begin{aligned} &\|v^2\|_{V_2(Q)}+\|\theta-\theta_\Gamma\|_{V_2(Q)}\leqslant K^{(6)},\\ &\|v^2(\cdot,t)\|_\Omega+\|\theta(\cdot,t)-\theta_\Gamma\|_\Omega\to 0 \quad \text{as } t\to\infty. \end{aligned} \tag{34}$$

Proof. By rewriting Eq. (23) as follows

$$\left(\frac{1}{2}v^2+c_V(\theta-\theta_\Gamma)\right)_t=(\sigma v+\pi)_x+p_0[u]v_x+gv$$

and taking $L^2(\Omega)$-inner product with $\frac{1}{2}v^2+c_V(\theta-\theta_\Gamma)$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\frac{1}{2}v^2+c_V(\theta-\theta_\Gamma)\right)^2dx\\ &\quad+\int_\Omega[(\nu\rho v_x-p[u,\theta])v+\kappa[u,\theta]\rho\theta_x](vv_x+c_V\theta_x)\,dx\\ &=\int_\Omega(p_0[u]v_x+gv)\left(\frac{1}{2}v^2+c_V(\theta-\theta_\Gamma)\right)dx\\ &\quad-p_\Gamma\left(v\left(\frac{1}{2}v^2+c_V(\theta-\theta_\Gamma)\right)\right)\Big|_{x=M}. \end{aligned} \tag{35}$$

We also take $L^2(\Omega)$-inner product of the second equation (2) with v^3:

$$\frac{1}{4}\frac{d}{dt}\int_\Omega v^4\,dx+3\int_\Omega(\nu\rho v_x-p[u,\theta])v^2v_x\,dx=\int_\Omega gv^3\,dx-p_\Gamma v^3|_{x=M}.$$

By summing up equality (35) and the latter one multiplied by a parameter $\delta\geqslant 1$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega\left[\left(\frac{1}{2}v^2+c_V(\theta-\theta_\Gamma)\right)^2+\frac{\delta}{2}v^4\right]dx\\ &\quad+\int_\Omega\left[(1+3\delta)\nu\rho v^2v_x^2\,dx+c_V\kappa[u,\theta]\rho\theta_x^2\right]dx=-\int_\Omega(\nu c_V+\kappa[u,\theta])\,\rho vv_x\theta_x\,dx\\ &\quad+\int_\Omega\left[p_0[u]v_x\left(\frac{1}{2}v^2+c_V(\theta-\theta_\Gamma)\right)+p[u,\theta]((1+3\delta)v^2v_x+c_Vv\theta_x)\right]dx\\ &\quad+\int_\Omega gv\left(\left(\frac{1}{2}+\delta\right)v^2+c_V(\theta-\theta_\Gamma)\right)dx\\ &\quad-p_\Gamma\left(v\left(\left(\frac{1}{2}+\delta\right)v^2+c_V(\theta-\theta_\Gamma)\right)\right)\Big|_{x=M}=:I_1+I_2+I_3+I_4. \end{aligned}$$

Let us estimate the summands in the last equality. First, by using the two-sided bounds $\underline{u} \leqslant u \leqslant \bar{u}$ and $\underline{\kappa} \leqslant \kappa \leqslant \bar{\kappa}$, we deduce

$$K_1^{-1}\left(\delta||vv_x||_\Omega^2 + ||\theta_x||_\Omega^2\right) \leqslant \int_\Omega \left[(1+3\delta)\nu\rho v^2 v_x^2 + c_V \kappa[u,\theta]\rho\theta_x^2\right] dx,$$

and

$$|I_1| \leqslant K_2||vv_x||_\Omega||\theta_x||_\Omega \leqslant \frac{K_2^2}{4\varepsilon}||vv_x||_\Omega^2 + \varepsilon||\theta_x||_\Omega^2 \quad \forall \varepsilon > 0.$$

Second, by using the estimates $|p_0[u]| \leqslant K_3$ and

$$|p[u,\theta]| = |p[u,\theta_\Gamma] + p_1[u](\theta - \theta_\Gamma)| \leqslant K_4(1 + |\theta - \theta_\Gamma|),$$

we have

$$\begin{aligned} |I_2| \leqslant & K_5\left[\int_\Omega \left(\delta v^2|v_x| + |v\theta_x|\right) dx + \int_\Omega |\theta - \theta_\Gamma|\left(|v_x| + \delta v^2|v_x| + |v\theta_x|\right) dx\right] \\ &=: K_5(I_{21} + I_{22}). \end{aligned}$$

Furthermore, the following estimates hold, for any $\varepsilon > 0$:

$$I_{21} \leqslant \delta||vv_x||_\Omega||v||_\Omega + ||v||_\Omega||\theta_x||_\Omega \leqslant \varepsilon(\delta||vv_x||_\Omega^2 + ||\theta_x||_\Omega^2) + \frac{\delta+1}{4\varepsilon}||v||_\Omega^2$$

and

$$\begin{aligned} I_{22} \leqslant & ||\theta - \theta_\Gamma||_{L^\infty(\Omega)}\left|\left|\frac{v_x}{\sqrt{\theta}}\right|\right|_\Omega ||\theta||_{L^1(\Omega)}^{1/2} \\ & + ||\theta - \theta_\Gamma||_\Omega ||v||_{L^\infty(\Omega)}(\delta||vv_x||_\Omega + ||\theta_x||_\Omega) \\ & \leqslant \varepsilon\left(\frac{\delta}{2}||vv_x||_\Omega^2 + ||\theta_x||_\Omega^2\right) + \frac{MK^{(1)}}{2\varepsilon}\left|\left|\frac{v_x}{\sqrt{\theta}}\right|\right|_\Omega^2 + \frac{\delta}{\varepsilon}||v||_{L^\infty(\Omega)}^2||\theta - \theta_\Gamma||_\Omega^2. \end{aligned}$$

Third, we obtain

$$\begin{aligned} |I_3| + |I_4| & \leqslant (||g||_{L^1(\Omega)} + p_\Gamma)||v||_{C(\bar{\Omega})} M^{1/2}||(1+2\delta)vv_x + c_V\theta_x||_\Omega \\ & \leqslant \varepsilon(\delta||vv_x||_\Omega^2 + ||\theta_x||_\Omega^2) + \frac{K_6\delta}{\varepsilon}||v||_{C(\bar{\Omega})}^2, \end{aligned}$$

where all the above quantities K_i, $1 \leqslant i \leqslant 6$, do not depend on δ and ε.

Now, by choosing $\varepsilon := K_7^{-1}$ small enough and then $\delta := K_8$ large enough, and setting

$$y := \int_\Omega \left[\left(\frac{1}{2} v^2 + c_V(\theta - \theta_\Gamma) \right)^2 + \frac{\delta}{2} v^4 \right] dx,$$

we get

$$\frac{dy}{dt} + K_9^{-1}(||vv_x||_\Omega^2 + ||\theta_x||_\Omega^2) \leqslant K_{10}(ay + h), \tag{36}$$

with $a := ||v||^2_{L^\infty(\Omega)}$ and $h := ||\frac{v_x}{\sqrt{\theta}}||^2_\Omega$ (see (31)); moreover

$$K_{11}^{-1}\left(\frac{1}{2}||v^2||_\Omega^2 + ||\theta - \theta_\Gamma||_\Omega^2\right) \leqslant y \leqslant K_{11}\left(\frac{1}{2}||v^2||_\Omega^2 + ||\theta - \theta_\Gamma||_\Omega^2\right).$$

Clearly

$$\frac{dy}{dt} + K_{12}^{-1} y \leqslant K_{10}(ay + h),$$

with $K_{12} := K_9 K_{11} M^2$. By Corollary 1 we have $||a||_{L^1(\mathbf{R}^+)} \leqslant K^{(1)} ||h||_{L^1(\mathbf{R}^+)} \leqslant (K^{(5)})^2$, therefore Lemma 6 implies

$$\sup_{t \geqslant 0} y(t) \leqslant K_{13}, \quad y(t) \to 0 \quad \text{as } t \to \infty.$$

By integrating inequality (36) over $\mathbf{R}^+$, we also obtain

$$K_9^{-1}(||vv_x||_Q^2 + ||\theta_x||_Q^2) \leqslant y(0) + K_{10}\left(||a||_{L^1(\mathbf{R}^+)} \sup_{t \geqslant 0} y + ||h||_{L^1(\mathbf{R}^+)} \right),$$

so that $||vv_x||_Q + ||\theta_x||_Q \leqslant K_{14}$, and the proof is completed. □

Let us now estimate v_x in $L^2(Q)$.

Lemma 8. *The following estimate holds*: $||v_x||_Q \leqslant K^{(7)}$.

Proof. By taking $L^2(\Omega)$-inner product of the second equation (2) with v, we get the equality (cf. (25))

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2} v^2 + p_S u - P[u, \theta_\Gamma] \right) dx + \int_\Omega v \rho v_x^2 \, dx = \int_\Omega p_1[u](\theta - \theta_\Gamma) \, v_x \, dx.$$

By integrating it over $(0, T)$ and exploiting the bounds $\underline{u} \leqslant u \leqslant \bar{u}$, we get

$$||v_x||^2_{Q_T} \leqslant K_1(1 + ||\theta - \theta_\Gamma||_{Q_T} ||v_x||_{Q_T}).$$

Thus $||v_x||_{Q_T} \leqslant K_1^{1/2} + K_1||\theta - \theta_\Gamma||_{Q_T} \leqslant K_1^{1/2} + K_1 M||\theta_x||_{Q_T}$, for any $T>0$, and the result follows from the previous lemma. $\square$

Now we establish additional properties of $p[u,\theta] - p_S$.

Lemma 9. *The following estimate together with stabilization property hold*:

$$||p[u,\theta] - p_S||_Q \leqslant K^{(8)}, \tag{37}$$

$$||p[u,\theta](\cdot,t) - p_S(\cdot)||_\Omega \to 0 \quad \textit{as } t \to \infty. \tag{38}$$

Proof. 1. Eq. (26) implies the following equality, for any $T>0$,

$$||p[u,\theta] - p_S||^2_{Q_T} + ||I^* v_t||^2_{Q_T} = ||\nu\rho v_x||^2_{Q_T} + 2\int_{Q_T} (p[u,\theta] - p_S)\, I^* v_t\, dx\, dt.$$

Elementary transformations and the bounds $\underline{u} \leqslant u \leqslant \bar{u}$ give

$$\begin{aligned}
&\int_{Q_T} (p[u,\theta] - p_S) I^* v_t\, dx\, dt \\
&\quad = \int_{Q_T} (p[u,\theta_\Gamma] - p_S) I^* v_t\, dx\, dt + \int_{Q_T} p_1[u](\theta - \theta_\Gamma)\, I^* v_t\, dx\, dt \\
&\quad = \int_\Omega (p[u,\theta_\Gamma] - p_S) I^* v\, dx\Big|_0^T - \int_{Q_T} p_u[u,\theta_\Gamma] u_t\, I^* v\, dx\, dt \\
&\qquad + \int_{Q_T} p_1[u](\theta - \theta_\Gamma)\, I^* v_t\, dx\, dt \\
&\quad \leqslant K_1(||v(\cdot,T)||_\Omega + ||v^0||_\Omega + ||v_x||_{Q_T}||v||_{Q_T} + ||\theta - \theta_\Gamma||_{Q_T}||I^* v_t||_{Q_T}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&||p[u,\theta] - p_S||^2_{Q_T} + \tfrac{1}{2}||I^* v_t||^2_{Q_T} \\
&\quad \leqslant \nu\underline{u}^{-2}||v_x||^2_{Q_T} + K_1(||v(\cdot,T)||_\Omega + ||v^0||_\Omega + M||v_x||^2_{Q_T}) + (K_1 M)^2||\theta_x||^2_{Q_T},
\end{aligned}$$

so estimate (37) follows from Lemmas 1, 7, and 8.

2. First, instead of property (38), let us prove that

$$||p[u,\theta_\Gamma](\cdot,t) - p_S(\cdot)||_\Omega \to 0 \quad \text{as } t \to \infty. \tag{39}$$

By using the estimates $\underline{u}\leqslant u$, (37), and $||\theta_x||_Q\leqslant K^{(6)}$, we have

$$||p[u,\theta_\Gamma]-p_S||_Q\leqslant||p[u,\theta]-p_S||_Q+||p_1[u]||_{L^\infty(Q)}||\theta-\theta_\Gamma||_Q\leqslant K_2. \tag{40}$$

Then also

$$\begin{aligned}&\int_0^\infty\left|\frac{d}{dt}||p[u,\theta_\Gamma]-p_S||^2_\Omega\right|dt\\ &\quad=2\int_0^\infty\left|\int_\Omega p_u[u,\theta_\Gamma]u_t(\,p[u,\theta_\Gamma]-p_S)\,dx\right|dt\\ &\quad\leqslant 2||p_u[u,\theta_\Gamma]||_{L^\infty(Q)}||v_x||_Q||p[u,\theta_\Gamma]-p_S||_Q\leqslant K_3.\end{aligned} \tag{41}$$

Estimates (40) and (41) imply property (39).

Finally, by the bounds $\underline{u}\leqslant u\leqslant\bar{u}$ and the stabilization property (34) we get

$$\begin{aligned}&|||p[u,\theta]-p_S||^2_\Omega-||p[u,\theta_\Gamma]-p_S||^2_\Omega|\\ &\leqslant[2M^{1/2}(||p[u,\theta_\Gamma]||_{L^\infty(\Omega)}+\bar{p}_S)\\ &\quad+||p_1[u]||_{L^\infty(\Omega)}||\theta-\theta_\Gamma||_\Omega]||p_1[u]||_{L^\infty(\Omega)}||\theta-\theta_\Gamma||_\Omega\\ &\leqslant K_4(1+||\theta-\theta_\Gamma||_\Omega)||\theta-\theta_\Gamma||_\Omega\to 0\quad\text{as } t\to\infty,\end{aligned}$$

so that (39) implies (38). □

To establish the pointwise stabilization of the specific volume $u(x,t)$ as $t\to\infty$, we need a modification of the Ball-Pego Lemma [23] concerning "almost autonomous" ordinary differential equations.

Lemma 10. *Let $f\in C(\mathbf{R})$ be such that, for a given constant f_S, there exists no interval (z_1,z_2) such that $f(z)\equiv f_S$ on (z_1,z_2). Let also $\alpha,\beta\in C(\mathbf{R}^+)$ be such that $\alpha(t)\to 0$ and $\beta(t)\to 0$ as $t\to\infty$, as well as $a\in L^1(\mathbf{R}^+)$.*

If a function y satisfies $\sup_{t\geqslant 0}|y(t)|<\infty$, $y\in W^{1,1}(0,T)$ for all $T>0$, and

$$\frac{dy}{dt}=f(y+\alpha)-f_S+a+\beta\quad\text{on } \mathbf{R}^+,$$

then

$$y(t)\to y_S\quad\text{as } t\to\infty,\quad\text{and}\quad f(y_S)=f_S.$$

The result remains valid if one sets $\beta=0$ and replaces the condition $a\in L^1(\mathbf{R}^+)$ by the following ones

$$|a|\leqslant|a_1|+|\beta_1|,\quad a,a_1,\beta_1\in C(\mathbf{R}^+),\qquad a_1\in L^1(\mathbf{R}^+),\ \beta_1(t)\to 0\quad\text{as } t\to\infty.$$

Proof. We set $A(t) := \int_t^\infty a(\tau)\, d\tau$ and, for $z := y + A$, we get

$$\frac{dz}{dt} = f(z + \tilde{\alpha}) - f_S + \beta, \tag{42}$$

where $\tilde{\alpha} := \alpha - A \in C(\mathbf{R}^+)$ and $\tilde{\alpha}(t) \to 0$ as $t \to \infty$. Note that $z \in C^1(\mathbf{R}^+)$, by virtue of Eq. (42), and

$$\sup_{t \geqslant 0} |z(t)| \leqslant \sup_{t \geqslant 0} |y(t)| + ||a||_{L^1(\mathbf{R}^+)}.$$

Suppose that $z_1 := \liminf_{t \to \infty} z(t) < z_2 := \limsup_{t \to \infty} z(t)$. Then for any $z_0 \in (z_1, z_2)$, there exist two increasing sequences $t_k^+ \to \infty$ and $t_k^- \to \infty$ such that

$$z(t_k^+) = z(t_k^-) = z_0, \quad \frac{dz}{dt}(t_k^+) \geqslant 0, \quad \frac{dz}{dt}(t_k^-) \leqslant 0.$$

Eq. (42) applied for $t = t_k^\pm$ as $k \to \infty$ implies that $f(z_0) - f_S = 0$. Thus by contradiction with the condition on f, $z_1 = z_2 = z_S := \lim_{t \to \infty} z(t)$.

By integrating Eq. (42) over the interval $(k-1, k)$ and passing to the limit as $k \to \infty$, we obtain: $f(z_S) - f_S = 0$. It remains to use the equality $\lim_{t \to \infty} y(t) = \lim_{t \to \infty} z(t)$ to obtain the required result.

To prove the last part of the lemma, it suffices to apply the decomposition

$$a = \tilde{a} + \tilde{\beta} \quad \text{with } \tilde{a} := \frac{a}{|a_1| + \tilde{\beta}_1}|a_1|, \quad \tilde{\beta} := \frac{a}{|a_1| + \tilde{\beta}_1}\tilde{\beta}_1,$$
$$\tilde{\beta}_1(t) := |\beta_1(t)| + \frac{1}{t+1};$$

here $\tilde{a} \in L^1(\mathbf{R}^+)$, $\tilde{\beta} \in C(\mathbf{R}^+)$, and $\tilde{\beta}(t) \to 0$ as $t \to \infty$ (since $|\tilde{a}| \leqslant |a_1|$ and $|\tilde{\beta}| \leqslant |\beta_1(t)| + \frac{1}{t+1}$). $\square$

Lemma 11. *Let condition* (13) *be satisfied. Then the following pointwise stabilization property holds for the specific volume u: there exists a function* $u_S \in L^\infty(\Omega)$ *satisfying* (14) *such that* (15) *holds.*

Proof. For any fixed $x \in \bar{\Omega}$, we rewrite equation (27) in the following form

$$\frac{dy}{dt} = f(y + \alpha) - p_S + p_1[u](\theta - \theta_\Gamma), \tag{43}$$

with $y := v \log u - \alpha$, $\alpha := -I^* v$, and $f(z) := p(\exp \frac{z}{v}, \theta_\Gamma)$. Property (13) yields the corresponding property of f in Lemma 10, for any $f_S = p_S(\cdot)$.

By using the bounds $\underline{u} \leqslant u \leqslant \bar{u}$ and the stabilization property (34) we get

$$\sup_{t \geqslant 0} |y(t)| \leqslant \nu \max\{|\log \underline{u}|, |\log \bar{u}|\} + M^{1/2} \|v\|_{L^{2,\infty}(Q)} \leqslant K_1,$$

$$|\alpha(t)| \leqslant M^{1/2} \|v(\cdot, t)\|_\Omega \to 0 \quad \text{as } t \to \infty.$$

We also have, by the Young inequality

$$\begin{aligned} &|p_1[u](\theta - \theta_\Gamma)| \\ &\leqslant \|p_1[u]\|_{L^\infty(Q)} \|\theta - \theta_\Gamma\|_{C(\bar{\Omega})} \leqslant K_2 \|\theta_x\|_\Omega^{1/2} \|\theta - \theta_\Gamma\|_\Omega^{1/2} \\ &\leqslant \|\theta_x\|_\Omega^2 + K_2^{4/3} \|\theta - \theta_\Gamma\|_\Omega^{2/3} =: a_1 + \beta_1. \end{aligned}$$

The functions $a(\cdot, t) \coloneqq p_1[u](\cdot, t)(\theta(\cdot, t) - \theta_\Gamma)$ and a_1, β_1 satisfy the conditions of the last part of Lemma 10 by virtue of Lemma 7 (together with the properties $u(\cdot, t), \theta(\cdot, t), \|\theta_x(\cdot, t)\|_\Omega \in C(\mathbf{R}^+)$). Thus by Lemma 10, there exists

$$\lim_{t \to \infty} y(t) = y_S, \quad \text{with} \quad f(y_S) = p_S,$$

i.e. $u(\cdot, t) \to u_S(\cdot) = \exp \frac{y_S}{\nu}$ as $t \to \infty$ and $p(u_S(\cdot), \theta_\Gamma) = p_S(\cdot)$. The bounds $\underline{u} \leqslant u \leqslant \bar{u}$ and the measurability of $u(\cdot, t)$ on Ω imply the bounds $\underline{u} \leqslant u_S \leqslant \bar{u}$ and the measurability of u_S on Ω. $\square$

Note that the Lebesgue dominated convergence theorem immediately gives that $\|u(\cdot, t) - u_S(\cdot)\|_{L^q(\Omega)} \to 0$ as $t \to \infty$, for any $q \in [1, \infty)$.

To prove the stabilization for v in $L^q(\Omega)$, we turn to the auxiliary linear parabolic problem

$$\begin{cases} w_t = (\mu w_x - \psi)_x + g \quad \text{in } Q, \\ w|_{x=0} = 0, \ (\mu w_x - \psi)|_{x=M} = -p_\Gamma(t), \ w|_{t=0} = w^0(x). \end{cases} \tag{44}$$

Suppose that $\mu \in L^\infty(Q_T)$ and $\mu_t \in L^2(Q_T)$ for any $T > 0$, with $0 < \underline{\mu} \leqslant \mu$ in Q. Suppose also that $\psi \in L^{2,\infty}(Q)$, $g \in L^{1,\infty}(Q)$, $p_\Gamma \in L^\infty(\mathbf{R}^+)$, and that $w^0 \in H^1(\Omega)$, with $w^0(0) = 0$. Set $|||w|||_q \coloneqq \|w\|_{L^{q,\infty}(Q)} + \|w\|_{L^{\infty,q}(Q)}$ to shorten the notation.

Lemma 12. *Let $w \in H^1(Q_T) \cap L^\infty(Q_T)$, for any $T > 0$, be a weak solution to problem* (44) *such that $|||w|||_2 < \infty$. Then, for any $q \in [2, \infty)$, the following estimate together with stabilization property hold*:

$$|||w|||_q \leqslant C[\|w^0\|_{L^q(\Omega)} + q(\|\psi\|_{L^{2,\infty}(Q)} + \|g\|_{L^{1,\infty}(Q)} + \|p_\Gamma\|_{L^\infty(\mathbf{R}^+)} + |||w|||_2)],$$

$$\|w(\cdot, t)\|_{L^q(\Omega)} \to 0 \quad \textit{as } t \to \infty,$$

where C depends only on $\underline{\mu}$ and M.

More general assertions of such kind (together with applications to barotropic fluid equations) were given in [32–34], and the lemma follows from them.

Lemma 13. *Let* $||v^0||_{L^q(\Omega)} \leqslant N$, *for some* $q \in (4, \infty)$. *The following estimate together with stabilization property hold*:

$$|||v|||_q \leqslant qK^{(9)},$$

$$||v(\cdot, t)||_{L^q(\Omega)} \to 0 \quad as\ t \to \infty,$$

where $K^{(9)}$ *is independent of* q.

Proof. We consider v as the solution to problem (44) with given $\mu := v\rho$, $\psi := p[u, \theta]$. By the bounds $\underline{u} \leqslant u \leqslant \bar{u}$ and Lemma 7, the following estimates

$$K_1^{-1} \leqslant \mu,$$

$$||\psi||_{L^{2,\infty}(Q)} \leqslant M^{1/2} ||p[u, \theta_\Gamma]||_{L^\infty(Q)} + ||p_1[u]||_{L^\infty(Q)} ||\theta - \theta_\Gamma||_{L^{2,\infty}(Q)} \leqslant K_2,$$

$$|||v|||_2 \leqslant K_3$$

are valid, and the result follows from the previous lemma. □

By collecting all the results of Lemmas 1, 3, 5, 7–9, 11, and 13, the proof of Theorem 1 is complete.

3.2. *Proof of Proposition 1*

Note that condition $N^{-1} \leqslant \underline{p}_S$ has been used above in Lemma 1, but not in Lemma 3.

Let us turn to the proof of Lemma 1 and suppose that in contrast to (16) we have

$$\bar{V} := \sup_{t \geqslant 0} V(t) < \infty. \tag{45}$$

Since $p_S u = \varepsilon u + (p_S - \varepsilon) u$ and the estimate holds

$$\left| \int_\Omega (p_S - \varepsilon) u \, dx \right| \leqslant (|p_\Gamma| + ||g||_{L^1(\Omega)} + \varepsilon) \bar{V} \quad \forall \varepsilon > 0,$$

we see that Lemma 1 remains valid and consequently Lemma 3 does. The quantities $K^{(1)} - K^{(3)}$ now depend on $\bar{V}$ as well.

We consider Eq. (43). By applying the operator I_0 to it and exploiting the bound $\underline{u} \leqslant u$, we get

$$v \log u \geqslant v \log u^0 - I^*(v - v_0) + I_0(p[u, \theta_\Gamma] - p_S) - K_1 I_0 \max\{\theta_\Gamma - \theta, 0\}$$

since $p_1[u] \leqslant K_1$. Let us introduce the set $E_t := \{x \in \bar{\Omega}\colon\ \theta(x,t) \leqslant \theta_\Gamma\}$. By exploiting the equality $\max\{\theta_\Gamma - \theta, 0\}_x = -\theta_x \chi(E_t)$ with $\chi(E_t)$ being the characteristic function of E_t, we get

$$||\max\{\theta_\Gamma - \theta(\cdot,t), 0\}||_{C(\bar{\Omega})} \leqslant ||\theta_x(\cdot,t)||_{L^1(E_t)} \leqslant \left|\left|\frac{\theta_\Gamma}{\theta}\,\theta_x(\cdot,t)\right|\right|_{L^1(E_t)}$$

$$\leqslant \left|\left|\frac{\theta_\Gamma}{\theta}\theta_x(\cdot,t)\right|\right|_{L^1(\Omega)} \leqslant \theta_\Gamma V^{1/2}\left|\left|\frac{\sqrt{\rho}}{\theta}\,\theta_x(\cdot,t)\right|\right|_{\Omega}.$$

By using estimates (22) and (45) we find

$$(I_0 \max\{\theta_\Gamma - \theta, 0\})(\cdot,t) \leqslant \theta_\Gamma \bar{V}^{1/2} K^{(2)} t^{1/2}.$$

This estimate together with (28) imply

$$v \log u \geqslant -\frac{1}{\varepsilon} K_2 - \varepsilon t + I_0(p[u,\theta_\Gamma] - p_S) \quad \forall \varepsilon \in (0,1). \tag{46}$$

Since now $\underline{p}_S < m(\theta_\Gamma)$, for some x_0 as well as for $\varepsilon_0 > 0$ and $\delta > 0$, both small enough, we have

$$p_S(x) \leqslant m(\theta_\Gamma) - \varepsilon_0 \quad \text{for } x \in [x_0, x_0 + \delta] \subset \bar{\Omega}.$$

By choosing $\varepsilon := \varepsilon_0/2$, inequality (46) gives

$$v \log u \geqslant \frac{1}{2}\,\varepsilon_0 t - \frac{2}{\varepsilon_0} K_2 \quad \text{on } [x_0, x_0+\delta] \times \bar{\mathbf{R}}^+.$$

But therefore

$$V(t) \geqslant K_3 \delta \exp\left(\frac{\varepsilon_0}{2v}\,t\right) \to \infty \quad \text{as } t \to \infty,$$

with $K_3 := \exp(-\frac{2}{v\varepsilon_0} K_2)$, which clearly contradicts (45). $\square$

3.3. Proof of Proposition 2

Suppose that in contrast to (17)

$$\sup_{t \geqslant 0}\left|\int_\Omega v(x,t)\,dx\right| \leqslant C_1 < \infty. \tag{47}$$

We set $u_0(t) := u(0,t)$, consider Eq. (27) for $x = 0$ and integrate it in t:

$$v \log u_0(t) = v \log u^0(0) + \int_\Omega (v^0(x) - v(x,t))\,dx + \int_0^t p(u_0(\tau), \theta_\Gamma)\,d\tau \tag{48}$$

since $\theta|_{x=0} = \theta_\Gamma$ and now $p_S(0) = 0$. Straightforwardly (see (9) and (47))

$$\left| \nu \log u^0(0) + \int_\Omega (v^0(x) - v(x,t))\, dx \right| \leqslant K_1 + C_1. \tag{49}$$

Let us set $b(t) := \int_0^t p(u_0(\tau), \theta_\Gamma)\, d\tau$. Since $p(u, \theta_\Gamma) > m(\theta_\Gamma) = 0$, the function b is increasing and positive on $\mathbf{R}^+$. Moreover the following property holds:

$$b(t) \to \infty \quad \text{as } t \to \infty. \tag{50}$$

Indeed if, in contrast to this property, $0 < b(t) \leqslant C_2$ on $\mathbf{R}^+$, then according to (48) and (49)

$$0 < u_0(t) \leqslant C_3 \quad \text{on } \bar{\mathbf{R}}^+.$$

This estimate implies $p(u_0(t), \theta_\Gamma) \geqslant \varepsilon_0 > 0$ on $\bar{\mathbf{R}}^+$ and thus $b(t) \geqslant \varepsilon_0 t$ on $\mathbf{R}^+$. The contradiction proves (50).

Property (18) immediately follows from (48)–(50).

Let us justify the last part of Proposition 2. By the conditions on p_S and $p(u, \theta_\Gamma)$, we can consider

$$0 \leqslant p_S u, \quad -P(u, \theta_\Gamma) = \int_u^\infty p(\zeta, \theta_\Gamma)\, d\zeta > 0.$$

Thus if we turn to the proof of Lemma 1, we see that it remains valid but only the first summand in (21) should be dropped. In particular $\|v\|_{L^{2,\infty}(Q)} \leqslant K^{(1)}$, consequently property (47) holds, and by the first part of the proof so property (18) does.

3.4. *Proof of Theorem 2*

First, notice that properties (19) and (20) imply the following inequalities, for $0 < \theta$ and $x \in \bar{\Omega}$,

$$p(u, \theta) - p_S(x) \geqslant 0 \quad \text{for } 0 < u \leqslant \check{u}, \tag{51}$$

$$p(u, \theta) - p_S(x) \leqslant 0 \quad \text{for } \hat{u} \leqslant u, \tag{52}$$

Consequently

$$P(\check{u}, \theta) - P(u, \theta) = \int_u^{\check{u}} p(\zeta, \theta)\, d\zeta \geqslant p_S(x)(\check{u} - u) \quad \text{for } 0 < u \leqslant \check{u},$$

$$P(u, \theta) - P(\hat{u}, \theta) = \int_{\hat{u}}^u p(\zeta, \theta)\, d\zeta \leqslant p_S(x)(u - \hat{u}) \quad \text{for } \hat{u} \leqslant u.$$

Thus it is easy to verify that, for all $u>0$ and $x\in\bar{\Omega}$,

$$p_S(x)u - P(u,\theta_\Gamma) \geqslant C_0 := \min\{\underline{p}_S\check{u}, \underline{p}_S\hat{u}\} - \max_{\check{u}\leqslant u\leqslant\hat{u}} P(u,\theta_\Gamma).$$

Next, by applying Eq. (25) together with the last estimate we see that Lemma 1 remains valid but only the first summand in (21) should be dropped. Further, Eq. (27) and Remark 3 (with $N_1 = 0$ and $\varepsilon_0 = 0$), by exploiting estimates (28) and (51), (52), lead to both the uniform lower bound $\underline{u}\leqslant u(x,t)$ and the upper one $u(x,t)\leqslant\bar{u}$ in Q. After that, in fact, the rest of the proof of Theorem 1 (i.e., of Lemmas 7–9, 11, and 13) remains almost unchanged.

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Appendix

This appendix is devoted to the proof of the existence of a regular weak solution to the problem (2), (7), (8).

Proposition A.1. *Suppose that either conditions* (5), (6), *and* $N^{-1}\leqslant\underline{p}_S$, *or* (19) *and* (20) *are valid. Suppose also that* $\kappa_{uu}\in C(\mathbf{R}^+\times\mathbf{R}^+)$ *and* $u^0, v^0, \theta^0\in H^1(\Omega)$, $g\in L^2(\Omega)$ *with*

$$\|u^0\|_{H^1(\Omega)} + \|v^0\|_{H^1(\Omega)} + \|\theta^0\|_{H^1(\Omega)} \leqslant N, \quad \|g\|_{L^2(\Omega)}\leqslant N,$$

$$N^{-1}\leqslant u^0, \quad N^{-1}\leqslant\theta^0, \quad v^0(0) = 0, \quad \theta^0(0) = \theta_\Gamma.$$

Then for any $T>0$, *the problem* (2), (7), (8) *admits a unique regular weak solution, and it satisfies the following estimates*:

$$\|u_x\|_{L^{2,\infty}(Q_T)} + \|u_t\|_{L^{2,\infty}(Q_T)} + \|v\|_{H^{2,1}(Q_T)} + \|\theta\|_{H^{2,1}(Q_T)} \leqslant K^{(10)}, \tag{A.1}$$

$$0<\underline{u}\leqslant u(x,t)\leqslant\bar{u}, \quad 0<\underline{\theta} := (K^{(11)})^{-1}\leqslant\theta(x,t) \quad \textit{in } \bar{Q}_T. \tag{A.2}$$

Hereafter, the quantities $K^{(i)}$ *and* K_i *may depend also on* T.

Proof. We shall exploit a priori estimates for the solutions given in Theorems 1 and 2 and derive additional ones in Q_T in six steps. We shall end by a nonstandard proof of a local (in time) existence theorem.

1. We set $w := v(\log u)_x - v$ and rewrite the second equation (2) as follows

$$w_t = (p_{0u}[u] + p_{1u}[u]\theta)u_x + p_1[u]\theta_x - g.$$

By taking $L^2(\Omega)$-inner product with w, using the formula $u_x = \frac{1}{v}u(w + v)$ and the bounds $\underline{u} \leqslant u \leqslant \bar{u}$, we obtain the inequality

$$\frac{d}{dt}||w||^2_\Omega \leqslant K_1[(1 + ||\theta||_{L^\infty(\Omega)})(||w||^2_\Omega + ||v||^2_\Omega) + ||\theta_x||^2_\Omega + ||g||^2_\Omega].$$

The estimates $||\theta||_{L^\infty(\Omega)} \leqslant \theta_\Gamma + \sqrt{M}||\theta_x||_\Omega$, $||\theta_x||_Q \leqslant K^{(6)}$, and $||v(\log u^0)_x - v^0||_\Omega \leqslant K_2$, together with the Gronwall Lemma imply the bound $||w||_{L^{2,\infty}(Q_T)} \leqslant K_3$ and therefore

$$||u_x||_{L^{2,\infty}(Q_T)} \leqslant K^{(12)}. \tag{A.3}$$

Consequently, the function ρ is a Hölder continuous one on $\bar{Q}_T$.

2. The function $\hat{v} := I^*v$ satisfies the nondivergent parabolic problem (see (26) and (7), (8))

$$\begin{cases} \hat{v}_t = v\rho\hat{v}_{xx} + p[u,\theta] - p_S & \text{in } Q, \\ \hat{v}_x|_{x=0} = 0, \ \hat{v}|_{x=M} = 0, \ \hat{v}|_{t=0} = I^*v^0(x), \end{cases} \tag{A.4}$$

cf. [8]. The standard parabolic $H^{2,1;q}(Q_T)$-estimates [20] together with the bounds $\underline{u} \leqslant u \leqslant \bar{u}$, $||\theta||_{L^6(Q_T)} \leqslant c||\theta||_{V_2(Q_T)} \leqslant K_1$ lead to the estimate

$$||v_x||_{L^6(Q_T)} = ||\hat{v}_{xx}||_{L^6(Q_T)} \leqslant K_2(||p[u,\theta] - p_S||_{L^6(Q_T)} + ||v^0||_{L^6(\Omega)}) \leqslant K^{(13)}. \tag{A.5}$$

3. We also can consider the second equation (2) as a linear parabolic equation

$$v_t = (v\rho v_x - p[u,\theta])_x + g,$$

with corresponding boundary and initial conditions (see (7) and (8)). After the bounds $\underline{u} \leqslant u \leqslant \bar{u}$, (A.3), and (A.5), we have $||\rho_x||_{L^{2,\infty}(Q_T)} \leqslant K_1$ and

$$||p[u,\theta]_x||_{Q_T} \leqslant K_2[(1 + ||\theta||_{L^{\infty,2}(Q_T)})||u_x||_{L^{2,\infty}(Q_T)} + ||\theta_x||_{Q_T}] \leqslant K_3,$$

$$||p[u,\theta]_t||_{Q_T} \leqslant K_4[(1 + ||\theta||_{L^4(Q_T)})||v_x||_{L^4(Q_T)} + ||\theta_t||_{Q_T}] \leqslant K_5(1 + ||\theta_t||_{Q_T}).$$

Thus the standard parabolic $H^{2,1}(Q_T)$-estimates [20] (or [4]) imply

$$\begin{aligned} ||v||_{H^{2,1}(Q_T)} &\leqslant K_6(||p[u,\theta]||_{H^1(Q_T)} + ||g||_\Omega + |p_\Gamma| + ||v^0||_{H^1(\Omega)}) \\ &\leqslant K_7(1 + ||\theta_t||_{Q_T}). \end{aligned} \tag{A.6}$$

4. Let us turn to estimates for θ. We rewrite Eq. (24) as a linear parabolic equation

$$c_V\theta_t = (A\theta_x)_x + F, \tag{A.7}$$

with $A := \kappa[u,\theta]\rho$ and $F := (v\rho v_x - p_1[u]\theta)v_x$. By the bounds $\underline{\kappa} \leqslant \kappa \leqslant \bar{\kappa}$ and $\underline{u} \leqslant u \leqslant \bar{u}$, we get $K_1^{-1} \leqslant A \leqslant K_1$ and

$$||F||_{Q_T} \leqslant K_2(||v_x||_{L^4(Q_T)} + ||\theta||_{L^4(Q_T)})||v_x||_{L^4(Q_T)} \leqslant K_3, \tag{A.8}$$

where the estimates $||\theta||_{L^4(Q_T)} \leqslant K_4$ and (A.5) are again taken into account. Now, the standard parabolic $L^\infty(Q_T)$-estimates [20] (or [3]) imply

$$||\theta||_{L^\infty(Q_T)} \leqslant K_5(||F||_{Q_T} + \theta_\Gamma + ||\theta^0||_{L^\infty(\Omega)}) \leqslant K^{(14)}. \tag{A.9}$$

5. Let us derive a uniform lower bound for θ. We divide Eq. (A.7) by $-\theta^2$ and transform it as follows:

$$\begin{aligned} c_V(\theta^{-1})_t &= (A(\theta^{-1})_x)_x - 2A\theta^{-3}\theta_x^2 \\ &\quad - \left(\sqrt{v\rho}v_x\theta^{-1} - \frac{1}{2}\sqrt{\frac{u}{v}}p_1[u]\right)^2 + \frac{u}{4v}(p_1[u])^2. \end{aligned} \tag{A.10}$$

Set $d := \max\{\theta^{-1} - \theta_\Gamma^{-1}, 0\}$ and note that $d|_{x=0} = 0$ and $A(\theta^{-1})_x|_{x=M} = 0$. Now we multiply Eq. (A.10) by qd^{q-1} with $q \geqslant 2$, integrate the result over Ω, apply the bounds $\underline{u} \leqslant u \leqslant \bar{u}$ and the Hölder inequality and obtain

$$c_V \frac{d}{dt}\int_\Omega d^q\,dx \leqslant q\int_\Omega \frac{u}{4v}(p_1[u])^2 d^{q-1}\,dx \leqslant qK_1\left(\int_\Omega d^q\,dx\right)^{\frac{q-1}{q}}.$$

By solving this differential inequality (for example see [33, Lemma 1.4]), we find

$$||d(\cdot,t)||_{L^q(\Omega)} \leqslant ||d^0||_{L^q(\Omega)} + K_1 t,$$

with $d^0 := \max\{(\theta^0)^{-1} - \theta_\Gamma^{-1}, 0\} \leqslant N$. Passing to the limit as $q \to \infty$ gives

$$||d||_{L^\infty(Q_T)} \leqslant N + K_1 T =: K_2.$$

This estimate together with $\theta^{-1} \leqslant d + \theta_\Gamma^{-1}$ imply

$$\underline{\theta} := (K^{(11)})^{-1} \leqslant \theta \quad \text{in } Q_T. \tag{A.11}$$

6. Let us prove $H^{2,1}(Q_T)$-bound for θ. We introduce the function $\mathscr{K}(u,\theta) := \int_{\theta_\Gamma}^{\theta} \frac{\kappa(u,\tilde{\theta})}{u}\,d\tilde{\theta}$ and note that $\mathscr{K}[u,\theta]|_{x=0} = 0$. By taking $L^2(Q_\tau)$-inner product of Eq. (A.7) with $\mathscr{K}[u,\theta]_t$ we obtain (cf. [18])

$$\begin{aligned} &\int_{Q_\tau}(c_V\theta_t\mathscr{K}[u,\theta]_t + \pi\mathscr{K}[u,\theta]_{xt})\,dx\,dt \\ &\quad = \int_{Q_\tau} F\cdot\mathscr{K}[u,\theta]_t\,dx\,dt \quad \text{for } 0 \leqslant \tau \leqslant T. \end{aligned} \tag{A.12}$$

The following formulas hold

$$\mathcal{K}[u,\theta]_t = \mathcal{K}_u[u,\theta]\, v_x + A\theta_t, \quad \mathcal{K}[u,\theta]_x = \mathcal{K}_u[u,\theta]\, u_x + \pi,$$

$$\mathcal{K}[u,\theta]_{xt} = (\mathcal{K}_{uu}[u,\theta]\, v_x + \mathcal{K}_{u\theta}[u,\theta]\, \theta_t)u_x + \mathcal{K}_u[u,\theta]\, v_{xx} + \pi_t.$$

By using the bounds $\underline{u} \leqslant u \leqslant \bar{u}$ together with $\underline{\theta} \leqslant \theta \leqslant K^{(14)}$ (see (A.9) and (A.11)) we have the uniform bounds

$$|\mathcal{K}_u[u,\theta]| + |\mathcal{K}_{uu}[u,\theta]| + |\mathcal{K}_{u\theta}[u,\theta]| \leqslant K_0.$$

Now from equality (A.12) it follows that

$$\begin{aligned} & K_1^{-1} \|\theta_t\|_{Q_\tau}^2 + \tfrac{1}{2} \|\pi\|_\Omega^2 \Big|_0^\tau \\ & \leqslant K_2 \int_{Q_\tau} [|\theta_t|\, |v_x| + |\pi|((|v_x| + |\theta_t|)|u_x| + |v_{xx}|) + |F|(|v_x| + |\theta_t|)]\, dx\, dt \\ & \leqslant K_2[\|\theta_t\|_{Q_T} \|v_x\|_{Q_T} + \|\pi\|_{L^{\infty,2}(Q_T)} (\|v_x\|_{Q_T} + \|\theta_t\|_{Q_T}) \|u_x\|_{L^{2,\infty}(Q_T)} \\ & \quad + \|\pi\|_{Q_T} \|v_{xx}\|_{Q_T} + \|F\|_{Q_T} (\|v_x\|_{Q_T} + \|\theta_t\|_{Q_T})]. \end{aligned}$$

Let us use the estimates $\|v_x\|_{Q_T} \leqslant K^{(6)}$, $\|\pi\|_{Q_T} \leqslant K_3$ as well as (A.3), (A.6), and (A.8), for u_x, v, and F, respectively. By applying also the estimate $\|\pi\|_{L^{\infty,2}(Q_T)} \leqslant \sqrt{2} \|\pi\|_{Q_T}^{1/2} \|\pi_x\|_{Q_T}^{1/2} \leqslant \sqrt{2K_3} \|\pi_x\|_{Q_T}^{1/2}$, we get

$$\|\theta_t\|_{Q_T}^2 + \|\pi\|_{L^{2,\infty}(Q_T)}^2 \leqslant K_4(1 + \|\pi_x\|_{Q_T} + \|\theta_t\|_{Q_T}).$$

By combining this estimate and the trivial one $\|\pi_x\|_{Q_T} \leqslant c_V \|\theta_t\|_{Q_T} + \|F\|_{Q_T}$ (see (A.7)), we obtain

$$\|\theta_t\|_{Q_T} + \|\pi\|_{V_2(Q_T)} \leqslant K_5.$$

In particular $\|\theta_x\|_{L^{2,\infty}(Q_T)} \leqslant K_6$ and $\|\pi\|_{L^{\infty,2}(Q_T)} \leqslant \sqrt{M} K_5$.

Therefore by using the formula

$$\theta_{xx} = (A^{-1}\pi)_x = (\tilde{\kappa}_u[u,\theta] u_x + \tilde{\kappa}_\theta[u,\theta]\theta_x)\pi + \tilde{\kappa}[u,\theta]\pi_x,$$

with $\tilde{\kappa}(u,\theta) := \frac{u}{\kappa(u,\theta)}$, we also get

$$\|\theta_{xx}\|_{Q_T} \leqslant K_7[(\|u_x\|_{L^{2,\infty}(Q_T)} + \|\theta_x\|_{L^{2,\infty}(Q_T)})\|\pi\|_{L^{\infty,2}(Q_T)} + \|\pi_x\|_{Q_T}] \leqslant K_8.$$

Thus the estimate $\|\theta\|_{H^{2,1}(Q_T)} \leqslant K^{(15)}$ is proved. Consequently $\|v\|_{H^{2,1}(Q_T)} \leqslant K^{(16)}$, see (A.6). This completes the proof of all a priori estimates (A.1) and (A.2).

It is not difficult to verify the uniqueness of a regular weak solution similarly to [6].

7. Now we briefly describe the proof of a local existence theorem. Let us fix the data satisfying the hypotheses and the additional conditions

$$p_{0uu}, p_{1uu} \in C(\mathbf{R}^+), \quad u^0_{xx}, g_x \in L^2(\Omega). \tag{A.13}$$

We define the Banach space $\mathbf{B}_\tau, 0<\tau\leqslant T$, of triples $z=(u,v,\theta)$ equipped with the norm $||z||_{\mathbf{B}_\tau} = ||z||_{Q_\tau} + ||z_x||_{L^4(Q_\tau)} + ||u_t||_{Q_\tau}$ and the bounded closed convex set

$$S_\tau = \{z\in\mathbf{B}_\tau\colon\ ||z_x||_{L^4(Q_\tau)} + ||u_t||_{Q_\tau} \leqslant N_1,\ (2N)^{-1}\leqslant u$$
$$\leqslant 2c_0N, (2N)^{-1}\leqslant\theta\leqslant 2c_0N, v|_{x=0}=0\},$$

where $N_1>0$ and c_0 is such that $u^0\leqslant c_0N,\ \theta^0\leqslant c_0N$.

We introduce also the nonlinear operator $\mathscr{A}: S_\tau\to\mathbf{B}_\tau$ such that $\mathscr{A}(\tilde u,\tilde v,\tilde\theta) = (u,v,\theta)$, where θ and v satisfy the linear parabolic equations

$$c_V\theta_t = (\kappa[\tilde u,\tilde\theta]\tilde\rho\theta_x)_x + (v\tilde\rho\tilde v_x - p_1[\tilde u]\tilde\theta)\tilde v_x \quad \text{in } Q_\tau, \tag{A.14}$$

$$v_t = (v\tilde\rho v_x - p[\tilde u,\theta])_x + g \quad \text{in } Q_\tau, \tag{A.15}$$

with $\tilde\rho = \tilde u^{-1}$, and $u>0$ satisfies the ordinary differential equation (with respect to t)

$$(v\log u)_t = p[u,\theta] - p_S - I^*v_t \quad \text{in } Q_\tau, \tag{A.16}$$

see (27), together with the boundary conditions

$$\theta|_{x=0} = \theta_\Gamma, \quad (\kappa[\tilde u,\tilde\theta]\tilde\rho\theta_x)|_{x=M} = 0, \tag{A.17}$$

$$v|_{x=0} = 0, \quad (v\tilde\rho v_x - p[\tilde u,\theta])|_{x=M} = -p_\Gamma, \tag{A.18}$$

and the initial conditions (8).

Problems (A.14) and (A.17); (A.15) and (A.18); and (A.16), with the initial conditions (8), can be solved sequentially. By the linear parabolic equation theory there exist unique solutions $\theta, v\in H^{2,1}(Q_\tau)$ to the first and second problems, and they satisfy the estimates

$$||\theta||_{H^{2,1}(Q_\tau)} \leqslant K_1\exp(K_2||(\kappa[\tilde u,\tilde\theta]\tilde\rho)_x||^4_{L^4(Q_\tau)})(1+||\tilde v_x||^2_{L^4(Q_\tau)}) \leqslant K_3, \tag{A.19}$$

$$||v||_{H^{2,1}(Q_\tau)} \leqslant K_4\exp(K_5(1+||\tilde\rho_x||^4_{L^4(Q_\tau)}))(1+||p[\tilde u,\theta]||_{H^1(Q_\tau)}) \leqslant K_6, \tag{A.20}$$

cf. above items 3 and 6. Hereafter the quantities K_i (excluding K_1, K_2 and K_4, K_5) depend also on N_1.

The following inequalities hold:

$$\|\varphi\|_{L^4(Q_\tau)} \leqslant c_1(M,T)\tau^{1/12}\|\varphi\|_{V_2(Q_\tau)} \quad \forall \varphi \in V_2(Q_\tau), \tag{A.21}$$

$$\|\varphi - \varphi|_{t=0}\|_{C(\bar{Q}_\tau)} \leqslant c_2(M)\tau^{1/4}\|\varphi\|_{H^{2,1}(Q_\tau)} \quad \forall \varphi \in H^{2,1}(Q_\tau) \tag{A.22}$$

(which follow from the Hölder inequality, the embedding $V_2(Q_T) \subset L^6(Q_T)$, and the inequality $\|\varphi\|_{C(\bar{\Omega})} \leqslant c_3(M)\|\varphi\|_\Omega^{1/2}\|\varphi\|_{H^1(\Omega)}^{1/2}$). Thus, for $0 < \tau \leqslant \tau_1$ small enough,

$$\|\theta_x\|_{L^4(Q_\tau)} + \|v_x\|_{L^4(Q_\tau)} \leqslant N_1/2, \quad (2N)^{-1} \leqslant \theta \leqslant 2c_0 N \quad \text{in } \bar{Q}_\tau. \tag{A.23}$$

We rewrite the problem for u as the following integral equation:

$$v \log u = v \log u^0 + I_0(p[u,\theta] - p_S) - I^*(v - v^0). \tag{A.24}$$

For $0 < \tau \leqslant \tau_2$ small enough, this equation has a unique solution $u \in C(\bar{Q}_\tau)$, $u > 0$, and it satisfies the bounds

$$(2N)^{-1} \leqslant u \leqslant 2c_0 N \quad \text{in } \bar{Q}_\tau. \tag{A.25}$$

Moreover, from (A.16) and (A.24) it follows that $u_t \in V_2(Q_\tau)$, $u \in H^{2,1}(Q_\tau)$, and

$$\|u_t\|_{V_2(Q_\tau)} \leqslant K_7, \quad \|u_x\|_{L^{2,\infty}(Q_\tau)} \leqslant K_8, \quad \|u_{xx}\|_{L^{2,\infty}(Q_\tau)} \leqslant K_9 \tag{A.26}$$

(for the last estimate we use conditions (A.13)). Therefore by applying inequality (A.21), for $0 < \tau \leqslant \tau_3$ small enough,

$$\|u_x\|_{L^4(Q_\tau)} + \|u_t\|_{Q_\tau} \leqslant N_1/2. \tag{A.27}$$

In addition, the following estimate holds:

$$\sup_{0<\gamma<\tau} \gamma^{-1/2}\|\Delta_\gamma u_t\|_{Q_{\tau-\gamma}} \leqslant K_{10} \tag{A.28}$$

with $\Delta_\gamma \varphi(x,t) = \varphi(x,t+\gamma) - \varphi(x,t)$. This estimate is valid by virtue of the equation

$$v(\log u)_t = p[u,\theta] - p[\tilde{u},\theta] + v\tilde{\rho}v_x$$

(where Eq. (A.15) and the right-hand boundary condition (A.18) are used) and the known estimate

$$\sup_{0<\gamma<\tau} \gamma^{-1/2}\|\Delta_\gamma \varphi_x\|_{Q_{\tau-\gamma}} \leqslant c_4(M,T)\|\varphi\|_{H^{2,1}(Q_\tau)} \quad \forall \varphi \in H^{2,1}(Q_\tau).$$

Thus, for $\bar{\tau} = \min\{\tau_1, \tau_3\}$, the operator $\mathscr{A}$ is well defined and $\mathscr{A}(S_{\bar{\tau}}) \subset S_{\bar{\tau}}$, see (A.23), (A.25), and (A.27). Moreover estimates (A.19), (A.20), (A.26), and (A.28) imply that the set $\mathscr{A}(S_{\bar{\tau}})$ is precompact in $\mathbf{B}_{\bar{\tau}}$.

To prove the continuity of $\mathscr{A}$, take a sequence $\{\tilde{z}_n\} \subset S_{\bar{\tau}}$, $||\tilde{z}_n - \tilde{z}||_{\mathbf{B}_{\bar{\tau}}} \to 0$ as $n \to \infty$ and set $z_n = (u_n, v_n, \theta_n) := \mathscr{A}\tilde{z}_n$ and $z = (u, v, \theta) := \mathscr{A}\tilde{z}$. By considering problems for $\theta - \theta_n$ and $v - v_n$, applying the standard parabolic energy estimate and estimates (A.19), (A.20), we obtain

$$||\theta - \theta_n||_{V_2(Q_{\bar{\tau}})} \leqslant K_{11} ||\tilde{z} - \tilde{z}_n||_{\mathbf{B}_{\bar{\tau}}} \to 0,$$

$$||v - v_n||_{V_2(Q_{\bar{\tau}})} \leqslant K_{12} (||\tilde{z} - \tilde{z}_n||_{\mathbf{B}_{\bar{\tau}}} + ||\theta - \theta_n||_{Q_{\bar{\tau}}}) \to 0.$$

By considering the difference of Eq. (A.15) for u and the similar one for u_n, we also obtain

$$||u - u_n||_{L^{2,\infty}(Q_{\bar{\tau}})} \leqslant K_{13} (||\theta - \theta_n||_{Q_{\bar{\tau}}} + ||v - v_n||_{L^{2,\infty}(Q_{\bar{\tau}})}) \to 0.$$

Since the set $\mathscr{A}(S_{\bar{\tau}})$ is precompact, the last three limiting properties imply that $||z - z_n||_{\mathbf{B}_{\bar{\tau}}} \to 0$.

By combining all the properties of $S_{\bar{\tau}}$ and $\mathscr{A}$ and applying the classical Schauder theorem, we establish that $\mathscr{A}$ has a fixed point in $S_{\bar{\tau}}$. Evidently this fixed point serves as a regular weak solution to the original problem (2), (7), (8) in $Q_{\bar{\tau}}$.

Conditions (A.13) can be removed by the standard argument (by smoothing p_0, p_1 and u^0, g and passing to the limit). $\square$

Remark 4. In the case $\kappa = \kappa(u)$, the existence of $\kappa_{uu} \in C(\mathbf{R}^+)$ is not required and the proof can be essentially simplified. Namely, the standard parabolic $H^{2,1}(Q_T)$-estimates imply $||\theta||_{H^{2,1}(Q_T)} \leqslant K^{(15)}$ in step 4, and estimate (A.9) in step 4 together with the main part of step 6 can be omitted.

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